§1.4 The Matrix Equation $Ax = b$

example from §1.3 we looked for a
olution to

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

we can write this in more compact notation

$$Ax = b$$

where $A = [v_1 \ \ v_2] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ \underbrace{4}_{v_1} & \underbrace{3}_{v_2} \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$

and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

with this we have <u>colum matrix multiplication</u>
(more on matrix multiplication in §2.1)

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + x_2 \\ 4x_1 + 3x_2 \end{bmatrix}$$ when set

equal to $b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$ we recover the
linear system

$$\begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 + x_2 = 5 \\ 4x_1 + 3x_2 = 9 \end{cases}$$

In general if $A = [v_1 \ v_2 \ \text{—} \ v_n]$ is an $m \times n$ matrix, i.e. each of $v_1, \cdots, v_n$ is an $m \times 1$ column, then the equation

$$Ax = b \qquad \text{where} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is a <u>matrix equation</u> equivalent to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

and is equivalent to the linear system whose augmented matrix is

$$A|b = [v_1 \ v_2 \ \cdots v_n | b]$$

<u>Remark</u>

$Ax = b$ has a solution if and only if b is a linear combination of $v_1, \cdots, v_n$ which is equivalent to saying b is in span$\{v_1, \cdots, v_n\}$

In this setting we say $Ax = b$ is a <u>consistent</u> matrix equation.

Example

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 5 \\ 3 & 2 & 2 \end{bmatrix}$ $\quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Is the equation $Ax = b$ consistent for all b_1, b_2, b_3?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 2 & 5 & b_2 \\ 3 & 2 & 2 & b_3 \end{array}\right] \xrightarrow{-3R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 2 & 5 & b_2 \\ 0 & -4 & -10 & -3b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 5/2 & \frac{1}{2}b_2 \\ 0 & -4 & -10 & -3b_1 + b_3 \end{array}\right] \xrightarrow[4R_2 + R_3 \to R_3]{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -b_2 + b_1 \\ 0 & 1 & 5/2 & \frac{1}{2}b_2 \\ 0 & 0 & 0 & -3b_1 + 2b_2 + b_3 \end{array}\right]$$

Answer: No!

$Ax = b$ has a solution if and only if

$$-3b_1 + 2b_2 + b_3 = 0$$

<u>Question</u>: What does it mean about A if $Ax = b$ has a solution (is consistent) for all b?

Defn

We say a set of vectors $\{v_1, \dots, v_n\}$ spans $\mathbb{R}^m$ if every vector b in $\mathbb{R}^m$ is in $\mathrm{span}\{v_1, \dots, v_n\}$. In other words every vector b in $\mathbb{R}^m$ is a linear combination of $v_1, \dots, v_n$. If this occurs, we write $\mathrm{Span}\{v_1, \dots, v_n\} = \mathbb{R}^m$.

Theorem

Let A be an $m \times n$ matrix. The following are equivalent:

a) For every b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution

b) Every b in $\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$

d) A has a pivot position in every row

Careful! We really mean A, not the augmented matrix $A|b$

Proof

(a) $\Longleftrightarrow$ (b) : We had seen before that for some b in $\mathbb{R}^m$, $Ax=b$ having a solution is equivalent to b being a linear combination of the columns of A. Now let b be any vector of $\mathbb{R}^m$.

if and only if symbol (equivalence)

(b) $\Longleftrightarrow$ (c) : This follows from the definition of what it means to span $\mathbb{R}^m$

"implies" symbol

(a) $\Rightarrow$ (d) : Suppose $Ax=b$ has a solution for all b in $\mathbb{R}^m$. Let E denote the reduced echelon form of A. Then after row reduction

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{c|c} E & c \end{array}\right]$$

for some c in $\mathbb{R}^m$. Since $Ax=b$ is consistent, by the theorem from (1/25), there is no pivot in c, i.e. there is no row of $E|c$ of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & c_m \end{array}\right]$$ where c_m is any real number.

Thus E has no rows of zeroes. Hence each row of E has a leftmost nonzero entry, which (by defn of reduced echelon form) is a leading 1 and hence a pivot.

Proof (cont.)

(d) $\Rightarrow$ (a): Suppose A has a pivot position in each row. Letting E denote the reduced echelon form of A and letting b be any vector in $\mathbb{R}^m$, after row reduction we can obtain

$$\left[\begin{array}{c|c} A & b \end{array}\right] \longrightarrow \left[\begin{array}{c|c} E & c \end{array}\right]$$

for some c in $\mathbb{R}^m$. Since A has a pivot in each row, so does E. Thus the column c is not a pivot column since pivot positions are the leftmost nonzero entries of $E|C$ and they all belong to E. Now by the theorem from (1/25), $Ax = b$ must have a solution.

Example

Do the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 1 \\ -8 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

Solution: Letting $A = \begin{bmatrix} 1 & 3 & -4 & 11 \\ 1 & 1 & -1 & 1 \\ 2 & 0 & 1 & -8 \end{bmatrix}$, by the previous theorem it suffices to check whether A has a pivot in every row or not.

$$A \xrightarrow[-2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & -4 & 11 \\ 0 & -2 & 3 & -10 \\ 0 & -6 & 9 & -30 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & -4 & 11 \\ 0 & 1 & -3/2 & 5 \\ 0 & -6 & 9 & -30 \end{bmatrix}$$

$$\xrightarrow[-3R_2+R_1 \to R_1]{6R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1/2 & -4 \\ 0 & 1 & -3/2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No! Since there is not a pivot position in every row.